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Dispersionless hierarchies, Hamilton–Jacobi theory and twistor correspondences ★

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Abstract

The dispersionless KP and Toda hierarchies possess an underlying twistorial structure. A twistorial approach is partly implemented by the method of Riemann–Hilbert problem. This is however still short of clarifying geometric ingredients of twistor theory such as twistor lines and twistor surfaces. A more geometric approach can be developed in a Hamilton–Jacobi formalism of Gibbons and Kodama. © 1998 Elsevier Science B.V.

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1. Introduction

The dispersionless KP and Toda hierarchies are the most typical dispersionless integrable hierarchies. The ordinary KP and Toda hierarchies have a Lax representation of the form

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad \text{etc.}, \tag{1}$$

where L, B_n, etc. are pseudo-differential or difference operators. Their dispersionless limit is a kind of "quasi-classical" limit in which L, B_n, etc. are phase space functions and the

★ Dedicated to Sir Roger Penrose on his 65th birthday.

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commutator [,] is replaced by a Poisson bracket { , }. This causes a drastic change, especially in the Toda hierarchy – the one-dimensional lattice in the Toda hierarchy turns into a continuous line. The dispersionless hierarchies nevertheless inherit many aspects of integrability from the original KP and Toda hierarchies, such as the notion of τ functions, Hirota equations, infinite-dimensional symmetries, etc. (For an overview, we refer to the review [26].)

In some aspects, however, the dispersionless hierarchies require an entirely new approach. One of them is the problem of constructing special solutions. A standard recipe is to consider a suitable "reduction" of the original infinite hierarchy into a system with a finite number of unknown functions. Usually, such a reduced system of the dispersionless hierarchies is a hydrodynamic system, and solvable by a generalized "hodographic" method as developed by Tsarev [27]. Many special solutions of the dispersionless KP and Toda hierarchies have been indeed constructed by Gibbons and Kodama [5,11,12] by a generalized hodographic method.

Proof of integrability of the dispersionless hierarchies themselves, too, has to be established by a new method. Such an approach is provided by the method of Riemann–Hilbert problem [22,23]. This Riemann–Hilbert problem is discovered as an analogue of the Riemann–Hilbert problem for the four-dimensional self-dual Einstein equation [1] and an associated infinite hierarchy [19]. In the case of the self-dual Einstein equation, the Riemann–Hilbert problem is an analytic expression of Penrose's curved twistor theory [18]. Thus the method of Riemann–Hilbert method may be called a twistorial approach to the dispersionless hierarchies. (For more details on this analogy, we refer to the review [20].)

Although thus nicely exhibiting a link with twistor theory, however, the previous approach by the Riemann–Hilbert problem [22,23] is still lacking a geometric language. Let us recall that a central idea of twistor theory is to relate a space–time manifold M with a complex manifold $\mathcal{T}$ (twistor space) by the "twistor correspondence", i.e., a manifold $\mathcal{F}$ with a double fibration $\pi_1 : \mathcal{F} \to M$ and $\pi_2 : \mathcal{F} \to \mathcal{T}$. By this correspondence, a space–time point $x \in M$ determines (and is determined by) a rational curve (twistor line) $\pi_2(\pi_1^{-1}(x))$ in the twistor space, and a twistor point $\xi \in \mathcal{T}$ a submanifold (twistor surface) $\pi_1(\pi_2^{-1}(\xi))$ of the space–time. These notions have been left obscure in the case of the dispersionless hierarchies.

The goal of this paper is to establish a dictionary between the twistor geometry and the method of Riemann–Hilbert problem for both the dispersionless KP and Toda hierarchy. It turns out that the Hamilton–Jacobi approach of Gibbons and Kodama [6,13], another proof of integrability of the dispersionless KP hierarchy, provides such a geometric interpretation of the method of Riemann–Hilbert problem. In Section 2, we review the previous results on the dispersionless KP hierarchy, and point out the problem. In Section 3, we reformulate the Hamilton–Jacobi approach of Gibbons and Kodama in our language, and present the twistorial interpretation. Section 4 is devoted to a similar treatment of the dispersionless Toda hierarchy. This part may be read as an independent result on an extension of the approach of Gibbons and Kodama. Section 5 is added for comments on a relation to Frobenius structures.

2. Dispersionless KP hierarchy and twistor theory

2.1. Lax formalism

Let us recall the Lax formalism of the dispersionless KP hierarchy [14,23].

Let (p, x) be canonical coordinates in a two-dimensional phase space with Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}. \tag{2}$$

Let $t = (t_2, t_3, \ldots)$ be a set of "time variables". The dispersionless KP hierarchy has the Lax representation

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \tag{3}$$

where $\mathcal{L}$ is a Laurent series of p of the form

$$\mathcal{L} = p + \sum_{n=1}^{\infty} u_{n+1}(x, t) p^{-n}, \tag{4}$$

$\mathcal{B}_n$ are the polynomial (in p) part of the nth power of $\mathcal{L}$,

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \tag{5}$$

and obey the Zakharov–Shabat equations

$$\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0. \tag{6}$$

One can extend these Lax equations by adding another Laurent series $\mathcal{M}$ of the form

$$\mathcal{M} = \sum_{n=2}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{n=1}^{\infty} v_n(x, t) \mathcal{L}^{-n-1} \tag{7}$$

that satisfies the Lax equations

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\} \tag{8}$$

and the canonical Poisson bracket relation

$$\{\mathcal{L}, \mathcal{M}\} = 1. \tag{9}$$

The dispersionless KP hierarchy is thus a "quasi-classical" limit of the ordinary KP hierarchy with commutators replaced by Poisson brackets. $\mathcal{L}$ corresponds to the Lax operator of the ordinary Lax formalism of the KP hierarchy. $\mathcal{M}$ is the dispersionless version of the Orlov–Schulman operator M [17].

2.2. Twistorial structure

The extended Lax formalism has another expression [23] which resembles twistor theory. Let ω be the 2-form

$$\omega = \mathrm{d}p \wedge \mathrm{d}x + \sum_{n=2}^{\infty} \mathrm{d}\mathcal{B}_n \wedge \mathrm{d}t_n. \tag{10}$$

In terms of this 2-form, the Zakharov–Shabat equations and the Lax equations can be rewritten in a very compact form as

$$\omega \wedge \omega = 0 \tag{11}$$

and

$$\mathrm{d}\mathcal{L} \wedge \mathrm{d}\mathcal{M} = \omega, \tag{12}$$

respectively. These equations show that ω is a degenerate symplectic form (on the infinite-dimensional (p, x, t) space) and that $\mathcal{L}$ and $\mathcal{M}$ are its Darboux variables. The 2-form ω is an analogue of Gindikin's "bundle of 2-form", which lies in the heart of his reformulation of curved twistor theory [7]. This suggests that the dispersionless KP hierarchy will give a kind of curved twistor theory.

This analogy is further deepened by the following characterization of $\mathcal{L}$ and $\mathcal{M}$ in terms of a Riemann–Hilbert problem [23]:

Theorem 1. *The functions $\mathcal{L} = \mathcal{L}(p, x, t)$ and $\mathcal{M} = \mathcal{M}(p, x, t)$ obey the functional equations (Riemann–Hilbert problem)*

$$\begin{aligned} \mathcal{L}(p,x,t) &= f(\bar{\mathcal{L}}(p,x,t), \bar{\mathcal{M}}(p,x,t)), \\ \mathcal{M}(p,x,t) &= g(\bar{\mathcal{L}}(p,x,t), \bar{\mathcal{M}}(p,x,t)), \end{aligned} \tag{13}$$

where $\bar{\mathcal{L}} = \bar{\mathcal{L}}(p, x, t)$ and $\bar{\mathcal{M}} = \bar{\mathcal{M}}(p, x, t)$ are a (unique) solution to the equations

$$\frac{\partial \bar{\mathcal{L}}}{\partial t_n} = \{\mathcal{B}_n, \bar{\mathcal{L}}\}, \quad \frac{\partial \bar{\mathcal{M}}}{\partial t_n} = \{\mathcal{B}_n, \bar{\mathcal{M}}\}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = 1, \tag{14}$$

under the initial condition

$$\bar{\mathcal{L}}(p, x, 0) = p, \quad \bar{\mathcal{M}}(p, x, 0) = x, \tag{15}$$

and $f = f(p, x)$ and $g = g(p, x)$ are given by

$$f(p, x) = \mathcal{L}(p, x, 0), \quad g(p, x) = \mathcal{M}(p, x, 0). \tag{16}$$

Proof. Note that f and g give a canonical pair

$$\{f, g\} = 1. \tag{17}$$

Therefore $\mathcal{L}$ and $f(\bar{\mathcal{L}}, \bar{\mathcal{M}})$ turn out to satisfy the same Lax equations. Furthermore they have an identical initial value at $t = 0$. Thus they coincide. Similarly, we find that $\mathcal{M} = g(\bar{\mathcal{L}}, \bar{\mathcal{M}})$. □

Let us specify why the above functional equations are interpreted as Riemann–Hilbert problem. In a suitable analytical setting, $\bar{\mathcal{L}}$ and $\bar{\mathcal{M}}$ are holomorphic functions of p in a neighbourhood $\bar{D}$ of $p = 0$ (because $\mathcal{B}_n$ are polynomials of p), whereas $\mathcal{L}$ and $\Xi = \mathcal{M} - \sum nt_n\mathcal{L}^{n-1}$ are holomorphic functions in a neighbourhood D of $p = \infty$. Suppose that D and $\bar{D}$ cover the whole Riemann sphere (e.g., this is the case if t is small). A precise meaning of the above functional equations is that these four functions $(\mathcal{L}, \Xi, \bar{\mathcal{L}}, \bar{\mathcal{M}})$ satisfy the equation

$$\mathcal{L} = f(\bar{\mathcal{L}}, \bar{\mathcal{M}}), \quad \Xi = g(\bar{\mathcal{L}}, \bar{\mathcal{M}}) - \sum_{n=2}^{\infty} nt_n f(\bar{\mathcal{L}}, \bar{\mathcal{M}})^{n-1} \tag{18}$$

over the intersection $D \cap \bar{D}$. This is a Riemann–Hilbert problem of the type that typically arise in curved twistor theory [1,18].

If the flows are restricted to the N-dimensional subspace $(x, t) = (x, t_1, \ldots, t_N, 0, \ldots)$, the analogy with curved twistor theory becomes more rigorous. One can indeed "twist" (f, g) into a map connecting $(\mathcal{L}^N, \mathcal{M}\mathcal{L}^{1-N}/N)$ and $(\bar{\mathcal{L}}, \bar{\mathcal{M}})$, use it as a transition function for constructing a mini-twistor space [8,10] by gluing two coordinate patches. The four functions then become "twistor functions".

In the full space of time evolutions (i.e., if $N = \infty$), the mini-twistor space ceases to exist. The Riemann–Hilbert problem itself is meaningful, but geometric ingredients of twistor theory, such as "twistor correspondence", "twistor lines" and "twistor surfaces" remain to be clarified. This is the problem that we are addressing.

Our claim is that the Hamilton–Jacobi approach of Gibbons and Kodama [6,13] may be used to resolve this problem.

3. Hamilton–Jacobi theory and twistor geometry

3.1. Multi-time Hamiltonian system

In this and next subsections, we reformulate the Hamilton–Jacobi approach of Gibbons and Kodama in our language. See also Carroll's paper [2] which presents an exposition of the Hamilton–Jacobi approach in a form more faithful to the formulation of Gibbons and Kodama.

Let λ and μ be parameters, and define two functions $p = p(\lambda, \mu, t)$ and $x = x(\lambda, \mu, t)$ implicitly by the equations

$$\mathcal{L}(p, x, t) = \lambda, \quad \mathcal{M}(p, x, t) = \mu. \tag{19}$$

Theorem 2. *$p = p(\lambda, \mu, t)$ and $x = x(\lambda, \mu, t)$ satisfy the multi-time Hamiltonian system*

$$\frac{\mathrm{d}p}{\mathrm{d}t_n} = \frac{\partial \mathcal{B}_n}{\partial x}, \quad \frac{\mathrm{d}x}{\mathrm{d}t_n} = -\frac{\partial \mathcal{B}_n}{\partial p} \tag{20}$$

with time-dependent Hamiltonians $\mathcal{B}_n = \mathcal{B}_n(p, x, t)$.

Proof. For avoiding complicated notations, let us write $p(\lambda, \mu, t)$ and $x(\lambda, \mu, t)$ as $p(t)$ and $x(t)$. By the definition, we have the identities

$$\mathcal{L}(p(t), x(t), t) = \lambda, \quad \mathcal{M}(p(t), x(t), t) = \mu. \tag{21}$$

Differentiating these identities in t_n gives

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t_n} + \frac{\partial \mathcal{L}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t_n} + \frac{\partial \mathcal{L}}{\partial t_n} = 0, \\ &\frac{\partial \mathcal{M}}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t_n} + \frac{\partial \mathcal{M}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t_n} + \frac{\partial \mathcal{M}}{\partial t_n} = 0. \end{aligned} \tag{22}$$

The Lax equations for $\mathcal{L}$ and $\mathcal{M}$ can be used here to evaluate the last terms in these equations. This gives

$$\begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial x} \\ \dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{M}}{\partial x} \end{pmatrix} \begin{pmatrix} \dfrac{\partial\, \mathrm{d}p}{\partial t_n} \\ \dfrac{\partial\, \mathrm{d}x}{\partial t_n} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial \mathcal{L}}{\partial t_n} \\ -\dfrac{\partial \mathcal{M}}{\partial t_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \mathcal{L}}{\partial p} & \dfrac{\partial \mathcal{L}}{\partial x} \\ \dfrac{\partial \mathcal{M}}{\partial p} & \dfrac{\partial \mathcal{M}}{\partial x} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathcal{B}_n}{\partial x} \\ -\dfrac{\partial \mathcal{B}_n}{\partial p} \end{pmatrix}. \tag{23}$$

The common 2×2 array is invertible (because of the canonical Poisson commutation relation of $\mathcal{L}$ and $\mathcal{M}$) and can be removed. The resulting equations are what we have sought for. □

The multi-time Hamiltonian system of Gibbons and Kodama can be thus reproduced from our extended Lax formalism.

This result can be restated in terms of canonical transformations. The Hamiltonian system is derived as equations of motion of a point (p, x) keeping $\mathcal{L}$ and $\mathcal{M}$ constant. In other words, $\mathcal{L}$ and $\mathcal{M}$ are invariants of the Hamiltonian flows:

$$\begin{aligned} \mathcal{L}(p(t), x(t), t) &= \mathcal{L}(p(0), x(0), 0), \\ \mathcal{M}(p(t), x(t), t) &= \mathcal{M}(p(0), x(0), 0). \end{aligned} \tag{24}$$

Meanwhile, by the canonical Poisson commutation relation between $\mathcal{L}$ and $\mathcal{M}$, the two-dimensional map $(p, x) \mapsto (\lambda, \mu) = (\mathcal{L}(p, x, t), \mathcal{M}(p, x, t))$ is symplectic. The above invariance property of $\mathcal{L}$ and $\mathcal{M}$ then implies that this symplectic map is a canonical transformation converting the multi-time Hamiltonian system to the Hamiltonian system

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t_n} = 0, \quad \frac{\mathrm{d}\mu}{\mathrm{d}t_n} = 0 \tag{25}$$

with *zero Hamiltonians*. This Hamiltonian system is further transformed into the Hamiltonian system

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t_n} = 0, \quad \frac{\mathrm{d}\xi}{\mathrm{d}t_n} = -n\lambda^{n-1} \tag{26}$$

by the simple transformation

$$\xi = \mu - \sum_{n=2}^{\infty} n t_n \lambda^{n-1}. \tag{27}$$

(Recall that the last transformation is just a disguise of the transformation $\Xi = \mathcal{M} - \sum nt_n\mathcal{L}^{n-1}$.) The last canonical variables (λ, ξ) are exactly those of Gibbons and Kodama. As they pointed out, the Hamiltonian system in these variables resembles the time evolution of *scattering data* in the conventional inverse scattering problem, which are also *action–angle variables*.

3.2. Generating function

Gibbons and Kodama formulated the above canonical transformation $(p, x) \mapsto (\lambda, \xi)$ in terms of a generating function. In our formulation, it is more convenient to consider the canonical transformation $(p, x) \mapsto (\lambda, \mu)$. Let $S(\lambda, x, t)$ be a generating function for the latter. The relations among the canonical variables (p, x), (λ, μ) and the Hamiltonian $\mathcal{B}_n$ are now written

$$\frac{\partial S(\lambda, x, t)}{\partial \lambda} = \mu, \qquad \frac{\partial S(\lambda, x, t)}{\partial x} = p, \qquad \frac{\partial S(\lambda, x, t)}{\partial t_n} = \mathcal{B}_n. \tag{28}$$

These relations can be cast into a more compact form:

$$\mathrm{d}S = \mu\,\mathrm{d}\lambda + p\,\mathrm{d}x + \sum_{n=2}^{\infty} \mathcal{B}_n\,\mathrm{d}t_n. \tag{29}$$

This shows that the generating function is nothing but the S-function that was discovered in an entirely different context [14,23]. The generating function of Gibbons and Kodama, say S_{GK}, is related to ours as $S_{\mathrm{GK}} = S - \sum \lambda^n t_n$.

The generating function $S(\lambda, x, t)$ is also related to the linear problem

$$\lambda\Psi = L\left(\hbar\frac{\partial}{\partial x}\right)\Psi, \quad \hbar\frac{\partial\Psi}{\partial\lambda} = M\left(\hbar\frac{\partial}{\partial x}\right)\Psi, \quad \hbar\frac{\partial\Psi}{\partial t_n} = B_n\left(\hbar\frac{\partial}{\partial x}\right)\Psi \tag{30}$$

of the KP hierarchy [24,26]. The WKB approximation

$$\Psi \sim \exp\hbar^{-1}S(\lambda, x, t) \tag{31}$$

to this *multi-time Schrödinger system* gives a system of Hamilton–Jacobi (or "eikonal") equations of the form

$$\begin{gathered} \lambda = \mathcal{L}\left(\frac{\partial S(\lambda, x, t)}{\partial x}\right), \qquad \frac{\partial S(\lambda, x, t)}{\partial\lambda} = \mathcal{M}\left(\frac{\partial S(\lambda, x, t)}{\partial x}\right), \\ \frac{\partial S(\lambda, x, t)}{\partial t_n} = \mathcal{B}_n\left(\frac{\partial S(\lambda, x, t)}{\partial x}\right). \end{gathered} \tag{32}$$

These Hamilton–Jacobi equations turn into the above defining relations of $S(\lambda, x, t)$ by the change of coordinates $(\lambda, x, t) \to (p, x, t)$ with

$$p = \frac{\partial S(\lambda, x, t)}{\partial x}. \tag{33}$$

Thus, the generating function $S(\lambda, x, t)$ links a *particle picture* (multi-time Hamiltonian system) and a *wave picture* (multi-time Schrödinger system).

3.3. Twistor geometry

Let us now compare these results with curved twistor theory. As the Riemann–Hilbert problem suggests, λ and μ may be thought of as coordinates on a (virtual) twistor space, and $\mathcal{L}$ and $\mathcal{M}$ as defining a twistor correspondence between space–time points and twistor points. According to the ordinary curved twistor theory of four-dimensional space–times [18], the level surfaces of $\mathcal{L}$ and $\mathcal{M}$ should be twistor surfaces. Actually, the present setting rather resembles the curved twistor theory of three-dimensional space–times [8,10]. From that point of view, it seems more suitable to interpret (x, t) as space–time coordinates and p as the fibre coordinate of a projectivized "spinor bundle". Accordingly, $p(t)$ is a "covariantly constant spinor field" evaluated at the space–time point $(x, t) = (x(t), t)$ of a twistor surface.

Summarizing, we have the following dictionary between the dispersionless KP hierarchy and the twistor geometry (in particular, the space–time side):
- $(\lambda, \mu) \longleftrightarrow$ mini-twistor space,
- $(x, t) \longleftrightarrow$ space–time,
- $p \longleftrightarrow$ fibre of projectivized spinor bundle,
- $(x(t), t) \longleftrightarrow$ twistor surface,
- $p(t) \longleftrightarrow$ covariantly constant spinor field on this twistor surface.

Of course the notion of mini-twistor space is only virtual, but the others have a definite meaning.

Having obtained this twistorial interpretation of the dispersionless KP hierarchy, we now turn to the dispersionless Toda hierarchy.

4. Dispersionless Toda hierarchy

4.1. Lax formalism

The Lax formalism of the dispersionless Toda hierarchy [22], too, is based on a two-dimensional phase space with coordinates (P, q). q is a continuum limit of the lattice coordinate, whereas P corresponds to the unit-shift operator. The Poisson bracket is given by

$$\{F, G\} = P\frac{\partial F}{\partial P}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}P\frac{\partial G}{\partial P}. \tag{34}$$

The corresponding 2-form is $\mathrm{d}\log P \wedge \mathrm{d}q$. In other words, $(\log P, q)$ gives a canonical pair.

The dispersionless Toda hierarchy consists of commuting flows of two Lax functions $\mathcal{L}$ and $\bar{\mathcal{L}}$ with time variables $(t = t_1, t_2, \ldots)$ and $(\bar{t} = \bar{t}_1, \bar{t}_2, \ldots)$. The Lax equations are given by

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{B_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_n} = \{\bar{B}_n, \mathcal{L}\}, \quad \frac{\partial \bar{\mathcal{L}}}{\partial t_n} = \{B_n, \bar{\mathcal{L}}\}, \quad \frac{\partial \bar{\mathcal{L}}}{\partial \bar{t}_n} = \{\bar{B}_n, \bar{\mathcal{L}}\}, \tag{35}$$

where Lax functions $\mathcal{L}$ and $\bar{\mathcal{L}}$ are two formal Laurent series of a variable P of the form

$$\mathcal{L} = P + \sum_{n=0}^{\infty} u_{n+1}(q,t,\bar{t})P^{-n}, \qquad \bar{\mathcal{L}} = \sum_{n=1}^{\infty} \bar{u}_n(q,t,\bar{t})P^{n}. \tag{36}$$

Furthermore, $\mathcal{B}_n$ and $\bar{\mathcal{B}}_n$ are given by

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}, \qquad \bar{\mathcal{B}}_n = (\bar{\mathcal{L}}^{-n})_{\leq -1}, \tag{37}$$

where $(\quad)_{\geq 0}$ and $(\quad)_{\leq -1}$ denote the projection of Laurent series of P into positive and negative powers respectively. They obey the Zakharov–Shabat equations

$$\begin{aligned} &\frac{\partial \mathcal{B}_m}{\partial t_n} - \frac{\partial \mathcal{B}_n}{\partial t_m} + \{\mathcal{B}_m, \mathcal{B}_n\} = 0, \\ &\frac{\partial \bar{\mathcal{B}}_m}{\partial \bar{t}_n} - \frac{\partial \bar{\mathcal{B}}_n}{\partial \bar{t}_m} + \{\bar{\mathcal{B}}_m, \bar{\mathcal{B}}_n\} = 0, \\ &\frac{\partial \mathcal{B}_m}{\partial \bar{t}_n} - \frac{\partial \bar{\mathcal{B}}_n}{\partial t_m} + \{\mathcal{B}_m, \bar{\mathcal{B}}_n\} = 0. \end{aligned} \tag{38}$$

This Lax formalism of the dispersionless Toda hierarchy can be extended by adding two counterparts $\mathcal{M}$ and $\bar{\mathcal{M}}$ of the dispersionless KP hierarchy. $\mathcal{M}$ and $\bar{\mathcal{M}}$ are Laurent series of the form

$$\begin{aligned} \mathcal{M} &= \sum_{n=1}^{\infty} n t_n \mathcal{L}^n + q + \sum_{n=1}^{\infty} v_n(q,t,\bar{t})\mathcal{L}^{-n}, \\ \bar{\mathcal{M}} &= -\sum_{n=1}^{\infty} n \bar{t}_n \bar{\mathcal{L}}^{-n} + q + \sum_{n=1}^{\infty} \bar{v}_n(q,t,\bar{t})\bar{\mathcal{L}}^{n} \end{aligned} \tag{39}$$

and satisfy the Lax equations

$$\begin{aligned} &\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}, \quad \frac{\partial \mathcal{M}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \mathcal{M}\}, \\ &\frac{\partial \bar{\mathcal{M}}}{\partial t_n} = \{\mathcal{B}_n, \bar{\mathcal{M}}\}, \quad \frac{\partial \bar{\mathcal{M}}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \bar{\mathcal{M}}\} \end{aligned} \tag{40}$$

and the canonical Poisson commutation relation

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}}. \tag{41}$$

4.2. Twistorial structure

The above extended Lax formalism, too, can be reformulated à la Gindikin [7]. Let ω be the 2-form

$$\omega = \mathrm{d}\log P \wedge \mathrm{d}q + \sum_{n=1}^{\infty} \mathrm{d}\mathcal{B}_n \wedge \mathrm{d}t_n + \sum_{n=1}^{\infty} \mathrm{d}\bar{\mathcal{B}}_n \wedge \mathrm{d}\bar{t}_n. \tag{42}$$

The Zakharov–Shabat equations then imply that

$$\omega \wedge \omega = 0, \tag{43}$$

and the Lax equations and the canonical Poisson commutation relations can be rewritten

$$\mathrm{d}\log\mathcal{L}\wedge \mathrm{d}\mathcal{M}=\omega=\mathrm{d}\log\bar{\mathcal{L}}\wedge \mathrm{d}\bar{\mathcal{M}}. \tag{44}$$

Thus ω is a denegerate symplectic form, and $(\mathcal{L},\mathcal{M})$ and $(\bar{\mathcal{L}},\bar{\mathcal{M}})$ are two different pairs of Darboux coordinates.

It is now straightforward to derive a Riemann–Hilbert problem [22]. The two pairs of Darboux coordinates are connected by functional relations of the form

$$\begin{aligned}\mathcal{L}(P,q,t,\bar{t}) &= f(\bar{\mathcal{L}}(P,q,t,\bar{t}),\bar{\mathcal{M}}(P,q,t,\bar{t}),t,\bar{t}),\\ \mathcal{M}(P,q,t,\bar{t}) &= g(\bar{\mathcal{L}}(P,q,t,\bar{t}),\bar{\mathcal{M}}(P,q,t,\bar{t}),t,\bar{t}).\end{aligned} \tag{45}$$

The transition functions $f=f(P,q)$ and $g=g(P,q)$ are required to satisfy the (twisted) canonical Poisson commutation relations

$$\{f(P,q),g(P,q)\}=f(P,q). \tag{46}$$

This gives the Riemann–Hilbert problem.

One can give a twistorial interpretation of this Riemann–Hilbert problem in exactly the same way as in the case of the dispersionless KP hierarchy. Note however that, unlike the case of the dispersionless KP hierarchy, $(\mathcal{L},\mathcal{M})$ and $(\bar{\mathcal{L}},\bar{\mathcal{M}})$ are both dynamical variables.

It should be also added that a twistorial formulation of the dispersionless Toda equation (the lowest member of the hierarchy) is presented by Ward [28] in the conventional mini-twistor language [8,10].

4.3. Multi-time Hamiltonian system

Because of the presense of two canonical variable pairs $(\mathcal{L},\mathcal{M})$ and $(\bar{\mathcal{L}},\bar{\mathcal{M}})$, the Hamilton–Jacobi formalism of the dispersionless Toda hierarchy is more complicated than the case of the dispersionless KP hierarchy.

Let (λ,μ) and $(\bar{\lambda},\bar{\mu})$ be two such pairs of parameters. One can define two functions $P=P(\lambda,\mu,t,\bar{t})$ and $q=q(\lambda,\mu,t,\bar{t})$ implicitly by

$$\mathcal{L}(P,q,t,\bar{t})=\lambda,\quad \mathcal{M}(P,q,t,\bar{t})=\mu, \tag{47}$$

and similarly, $P=\bar{P}(\bar{\lambda},\mu,t,\bar{t})$ and $q=\bar{q}(\bar{\lambda},\bar{\mu},t,\bar{t})$ by

$$\bar{\mathcal{L}}(P,q,t,\bar{t})=\bar{\lambda},\quad \bar{\mathcal{M}}(P,q,t,\bar{t})=\bar{\mu}. \tag{48}$$

We now have the following result:

Theorem 3. *Both* $(P(\lambda,\mu,t,\bar{t}),q(\lambda,\mu,t,\bar{t}))$ *and* $(\bar{P}(\bar{\lambda},\mu,t,\bar{t}),\bar{q}(\bar{\lambda},\bar{\mu},t,\bar{t}))$ *satisfy the same multi-time Hamiltonian system*

$$\frac{\mathrm{d}P}{\mathrm{d}t_n}=P\frac{\partial\mathcal{B}_n}{\partial q},\quad \frac{\mathrm{d}q}{\mathrm{d}t_n}=-P\frac{\partial\mathcal{B}_n}{\partial P},\quad \frac{\mathrm{d}P}{\mathrm{d}\bar{t}_n}=P\frac{\partial\bar{\mathcal{B}}_n}{\partial q},\quad \frac{\mathrm{d}q}{\mathrm{d}\bar{t}_n}=-P\frac{\partial\bar{\mathcal{B}}_n}{\partial P}. \tag{49}$$

Proof. The proof is almost the same as the case of the dispersionless KP hierarchy. The only difference is the Poisson bracket. (It will be more convenient to do calculations in the canonical pair $p = \log P$ and q rather than in P and q.) □

Since this Hamiltonian system lives on a two-dimensional phase space, its trajectories should form just a two-dimensional family. Accordingly, the two sets of parameters (λ, μ) and $(\bar{\lambda}, \bar{\mu})$ should be functionally related. Let us write the functional relations as

$$\lambda = f(\bar{\lambda}, \bar{\mu}), \quad \mu = g(\bar{\lambda}, \bar{\mu}). \tag{50}$$

This implies that the four functions $(\mathcal{L}, \mathcal{M}, \bar{\mathcal{L}}, \bar{\mathcal{M}})$ satisfy the functional relations

$$\begin{aligned} &\mathcal{L}(P, q, t, \bar{t}) = f(\bar{\mathcal{L}}(P, q, t, \bar{t}), \bar{\mathcal{M}}(P, q, t, \bar{t})), \\ &\mathcal{M}(P, q, t, \bar{t}) = g(\bar{\mathcal{L}}(P, q, t, \bar{t}), \bar{\mathcal{M}}(P, q, t, \bar{t})). \end{aligned} \tag{51}$$

Thus the Riemann–Hilbert problem of Section 4.2 can be reproduced from the multi-time Hamiltonian system.

Twistorial interpretation of these results is quite parallel to the case of the dispersionless KP hierarchy.

4.4. *Generating functions*

The two different parametrizations of trajectories of the multi-time Hamiltonian system lead to two different canonical transformations and multi-time Hamiltonian systems with zero Hamiltonians. The first canonical transformation $(P, q) \mapsto (\lambda, \mu)$ is defined by a generating function $S(\lambda, q, t, \bar{t})$ as

$$\frac{\partial S}{\partial \lambda} = \mu, \quad \frac{\partial S}{\partial q} = \log P, \quad \frac{\partial S}{\partial t_n} = \mathcal{B}_n, \quad \frac{\partial S}{\partial \bar{t}_n} = \bar{\mathcal{B}}_n. \tag{52}$$

The transformed Hamiltonian system is given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t_n} = 0, \quad \frac{\mathrm{d}\mu}{\mathrm{d}t_n} = 0. \tag{53}$$

Similarly, the second canonical transformation $(P, q) \mapsto (\bar{\lambda}, \bar{\mu})$ is defined by a generating function $\bar{S} = \bar{S}(\bar{\lambda}, q, t, \bar{t})$ as

$$\frac{\partial \bar{S}}{\partial \bar{\lambda}} = \bar{\mu}, \quad \frac{\partial \bar{S}}{\partial q} = \log P, \quad \frac{\partial \bar{S}}{\partial t_n} = \mathcal{B}_n, \quad \frac{\partial \bar{S}}{\partial \bar{t}_n} = \bar{\mathcal{B}}_n. \tag{54}$$

The transformed Hamiltonian system is given by

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}t_n} = 0, \quad \frac{\mathrm{d}\bar{\mu}}{\mathrm{d}t_n} = 0. \tag{55}$$

These Hamiltonian system can be further mapped to a system of Gibbons and Kodama type by a simple change of variables $(\lambda, \mu) \mapsto (\lambda, \xi)$ and $(\bar{\lambda}, \bar{\mu}) \mapsto (\bar{\lambda}, \bar{\xi})$ as

$$\xi = \mu - \sum_{n=1}^{\infty} n t_n \lambda^n, \qquad \bar{\xi} = \bar{\mu} + \sum_{n=1}^{\infty} n \bar{t}_n \bar{\lambda}^{-n}. \tag{56}$$

The defining equations of the canonical transformations can be rewritten

$$\begin{aligned} \mathrm{d}S &= \mu\,\mathrm{d}\lambda + q + \sum_{n=1}^{\infty} \mathcal{B}_n\,\mathrm{d}t_n + \sum_{n=1}^{\infty} \bar{\mathcal{B}}_n\,\mathrm{d}\bar{t}_n, \\ \mathrm{d}\bar{S} &= \bar{\mu}\,\mathrm{d}\bar{\lambda} + q + \sum_{n=1}^{\infty} \mathcal{B}_n\,\mathrm{d}t_n + \sum_{n=1}^{\infty} \bar{\mathcal{B}}_n\,\mathrm{d}\bar{t}_n. \end{aligned} \tag{57}$$

Thus S and $\bar{S}$ are, in fact, the same as those which are already known [22] and connected with the linear problem of the Toda lattice hierarchy [25,26].

5. Concluding remarks

The dispersionless KP and Toda hierarchies are also related to another geometric structure – Frobenius manifold.

Let us recall the notion of Frobenius algebra and Frobenius manifold. For more details, we refer to lecture notes of Dubrovin [4] and Hitchin [9].

A Frobenius algebra is a finite-dimensional commutative and associative algebra V with an identity element e and a linear form $\theta \in V^*$ for which $g(a,b) = \theta(ab)$, $a, b \in V$, is a nondegenerate inner product on V. This determines a symmetric form $g \in S^2V^*$. Furthermore, the multiplicative structure $V \otimes V \to V$ determines an element of $V \otimes V \otimes V^*$. Identifying $V^* \simeq V$ by the nondegenerate inner product, one obtains an element $c \in V^* \otimes V^* \otimes V^*$. By commutativity c becomes totally symmetric, i.e., $c \in S^3V^*$. The three data (θ, g, c) conversely determine a Frobenius algebra.

Let M be an n-manifold with a smoothly varying structure of Frobenius algebra in the tangent space at each point. This amounts to giving the following three data:
- $\theta \in C^\infty(T^*M)$,
- $g \in C^\infty(S^2T^*M)$,
- $c \in C^\infty(S^3T^*M)$.

Let e denote the identity element of the pointwise defined Frobenius algebra, and ∇ the covariant derivative of the Levi-Civita connection determined by the metric g. e is a vector field on M and called the Euler vector field. A manifold M with these data is called a Frobenius manifold if the following conditions are satisfied:
1. (M, g) is flat,
2. the Euler vector field e is covariantly constant,
3. ∇c is symmetric.

The flatness of the metric implies the existence of special local coordinates ("flat coordinates") t^i for which the coefficents of metric are constant:

$$g = \sum \eta_{ij}\,\mathrm{d}t^i\,\mathrm{d}t^j, \quad \eta_{ij} = \text{constant}. \tag{58}$$

The third condition above, meanwhile, implies the existence of another set of distinguished coordinates ("canonical coordinates"). They are local orthogonal coordinates u^i for which the metric is written in the following special diagonal form with a potential ϕ:

$$g = \sum \mu_i (\mathrm{d}u^i)^2, \quad \mu_i = \frac{\partial \phi}{\partial u^i}. \tag{59}$$

(A metric of this form is called, in general, an Egorov metric.)

A particularly important class of Frobenius manifolds are those with homogeneity. This means that all structure functions are homogeneous with respect to the Euler vector field e.

Dubrovin [3] pointed out that the moduli spaces of marked Riemann surfaces of arbitrary genus and with an arbitrary number of marked points ("Hurwitz spaces") have a Frobenius structure (with homogeneity). Two special classes of those Frobenius manifolds emerge as solutions of the dispersionless KP and Toda hierarchies. (Furthermore, these are examples where the Riemann–Hilbert problem can be solved rather explicitly.)

The first example is related to the dispersionless KP hierarchy. Let M be the space of complex polynomials

$$E(p) = p^{n+1} + a_1 p^{n-1} + \cdots + a_n. \tag{60}$$

This is an affine space and its tangent vectors at the point is represented by a polynomial $\dot{E}(p)$ of the form

$$\dot{E}(p) = \dot{E}_1 p^{n-1} + \cdots + \dot{E}_n. \tag{61}$$

We define a metric on M by

$$g(\dot{E}_1, \dot{E}_2) = \operatorname*{res}_{p=\infty} \frac{\dot{E}_1 \dot{E}_2}{E'}\,\mathrm{d}p = -\sum_i \operatorname*{res}_{p=\alpha_i} \frac{\dot{E}_1 \dot{E}_2}{E'}\,\mathrm{d}p, \tag{62}$$

where α_i's are the roots of $E'(p) = \mathrm{d}E(p)/\mathrm{d}p$, and the summation is over all those roots. M becomes a Frobenius manifold with canonical coordinates $u^i = E(\alpha_i)$.

In the context of the dispersionless KP hierarchy, $E(p)$ is related to $\mathcal{L}$ as

$$E(p) = \mathcal{L}^{n+1}. \tag{63}$$

This gives a "reduction" of the dispersionless KP hierarchy [14], which has only a finite number of unknown functions ($a_1, \ldots, a_{n-1}$ and a_n in the above notation). Solving this relation for p yields a Laurent series of the form

$$p = \mathcal{L} + b_1 \mathcal{L}^{-1} + b_2 \mathcal{L}^{-2} + \cdots. \tag{64}$$

The first n coefficients $b_1, \ldots, b_n$ then give flat coordinates.

Similarly, the family of trigonometric polynomials of the form

$$E(p) = e^{np} + a_1 e^{(n-1)p} + \cdots + a_n + a_{n+1} e^{-p} \tag{65}$$

gives another example of Frobenius manifold. If we identify $P = e^p$, this corresponds to a reduction of the dispersionless Toda hierarchy defined by the relations [21]

$$\mathcal{L}^n = E(p) = \bar{\mathcal{L}}^{-1}. \tag{66}$$

A very intriguing open problem is to interpret the "unreduced" dispersionless KP and Toda hierarchies as a kind of infinite-dimensional Frobenius manifolds. This will require a drastic modification of the notion of Frobenius manifold. This situation is reminiscent of our twistorial interpretation of these hierarchies – the dispersionless hierarchies have no twistor space in the ordinary sense, but still retain a virtual counterpart of twistor space and twistor lines (encoded in the Riemann–Hilbert problem) as well as the notion of twistor functions and twistor lines (related to a multi-time Hamiltonian system).

The same problem may be raised to Krichever's "universal Whitham hierarchy" [15], which includes all Hurwitz spaces of arbitrary genus and number of marked points as special solutions.

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